DIRECT SYSTEMS OF SPHERICAL FUNCTIONS AND REPRESENTATIONS

MATTHEW DAWSON, GESTUR ÓLAFSSON, AND JOSEPH A. WOLF

Abstract. Spherical representations and functions are the building blocks for harmonic analysis on riemannian symmetric spaces. Here we consider spherical functions and spherical representations related to certain infinite dimensional symmetric spaces $G_\infty/K_\infty = \varinjlim G_n/K_n$. We use the representation theoretic construction $\varphi(x) = \langle e, \pi(x)e\rangle$ where e is a K_∞–fixed unit vector for π. Specifically, we look at representations $\pi_\infty = \varinjlim \pi_n$ of G_∞ where π_n is K_n–spherical, so the spherical representations π_n and the corresponding spherical functions φ_n are related by $\varphi_n(x) = \langle e_n, \pi_n(x)e_n\rangle$ where e_n is a K_n–fixed unit vector for π_n, and we consider the possibility of constructing a K_∞–spherical function $\varphi_\infty = \lim \varphi_n$. We settle that matter by proving the equivalence of (i) $\{e_n\}$ converges to a nonzero K_∞–fixed vector e, and (ii) G_∞/K_∞ has finite symmetric space rank (equivalently, it is the Grassmann manifold of p–planes in $\mathbb{F}^\infty$ where $p < \infty$ and $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). In that finite rank case we also prove the functional equation

$$\varphi(x)\varphi(y) = \lim_{n\to\infty} \int_{K_n} \varphi(xky)dk$$

of Faraut and Olshanskii, which is their definition of spherical functions. We use this, and recent results of M. Rösler, T. Koornwinder and M. Voit, to show that in the case of finite rank all K_∞-spherical representations of G_∞ are given by the above limit formula. This in particular shows that the characterization of the spherical representations in terms of highest weights is still valid as in the finite dimensional case.

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1. Introduction

Representation theory and harmonic analysis on symmetric spaces is by now well understood. The building blocks are the spherical representations and the corresponding spherical functions. For the case of a compact symmetric space G/K the spherical representations are parameterized by a certain semi-lattice Λ. When G/K is simply connected, Λ is described by the Cartan-Helgason theorem. For each $\mu \in \Lambda$ let (π_μ, V_μ) denote the corresponding irreducible representation. Then the space V_λ^K of K-fixed vectors has dimension 1. Let e_μ be a unit vector in V_μ^K. The function

$$\psi_\mu(x) = \langle e_\mu, \pi_\mu(x)e_\mu\rangle \tag{1.1}$$

does not depend on the choice of e_μ , and $\{\sqrt{\dim V_\mu}\psi_\mu\}_{\mu\in\Lambda}$ is an orthonormal basis for $L^2(G/K)^K$. In particular

$$f = \sum_{\mu\in\Lambda} \dim V_\mu \langle f, \psi_\mu\rangle \psi_\mu$$

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for every $f \in L^2(G/K)^K$. Here the sum is taken in the L^2-sense. Similarly, every $f \in L^2(G/K)$ can be written as

$$f = \sum_{\mu \in \Lambda} \dim V_\mu \langle \pi_\mu(f) e_\mu, \pi_\mu(\,\cdot\,) e_\mu \rangle$$

where $\pi_\mu(f) = \int_G f(gK)\pi_\mu(g)\, d(gK)$.

The function ψ_μ is spherical in the sense that

$$\int_K \psi_\mu(xky)\, dk = \psi_\mu(x)\psi_\mu(y) \quad \text{for all } x, y \in G\,. \tag{1.2}$$

Here dk is normalized Haar measure on the compact group K. Every positive definite spherical function on G is obtained in this way from an irreducible unitary representation of G.

It is natural to extend the study to infinite dimensional Lie groups and symmetric spaces. The simplest case is $G_\infty = \varinjlim G_n$, $K_\infty = \varinjlim K_n$ and $M_\infty = \varinjlim G_n/K_n$ where $G_n \subseteq G_{n+1}$ is a sequence of compact Lie groups such that $K_n = G_n \cap K_{n+1}$. The basic theory was developed by G. Olshanskii (see [8]) for the classical direct limits and for a very important class of representations; by L. Natarajan, E. Rodrıguez–Carrington and one of us for more general direct limits (see [5], [6] and [7]); and by S. Strătilă & D. Voiculescu (see their survey [14]) for the factor representation viewpoint. See J. Faraut [2] for further information and references.

The equation (1.2) does not make sense here because there is no invariant measure on K_∞. The replacement is the functional equation

$$\lim_{n\to\infty} \int_{K_n} \psi(xk_ny)\, dk_n = \psi(x)\psi(y) \quad \text{for all } x, y \in G_\infty\,. \tag{1.3}$$

Again, see [2], the function ψ is spherical if and only if there is an irreducible unitary representation (π, V) of G_∞ and $e \in V^{K_\infty}$ with $\|e\| = 1$, such that ψ is given by (1.1).

On the other hand, limits of irreducible spherical representations for a strict direct system $\{M_n = G_n/K_n\}$ of compact symmetric spaces were studied by the last named author in a series of articles [15], [16], and [17], and then later by the last two authors in [10] and [11]. In particular, in [10], [11] and [12] they introduced the notion of propagation of symmetric spaces. In short, if the G_n are compact and connected, and π_n is a spherical representation of G_n, then there exists in a canonical way a spherical representation π_{n+1} of G_{n+1} such that π_n is a subrepresentation of $\pi_{n+1}|_{G_n}$ with multiplicity 1. Furthermore, if u_{n+1} is a highest weight vector for π_{n+1} then π_n is realized as $\pi_{n+1}|_{G_n}$ acting on the space generated by $\pi_{n+1}(G_n)u_{n+1}$. The system $\{(\pi_n, V_n)\}$ is injective and $(\pi_\infty, V_\infty) := \varinjlim(\pi_n, V_n)$ is an irreducible unitary representation of G_∞.

In this article we address the question of whether the representation (π_∞, V_∞) is spherical. Our main result is Theorem 4.5 below. It states that $V_\infty^{K_\infty} \neq \{0\}$ if and only if the symmetric space ranks of the compact riemannian symmetric spaces M_n are bounded. Thus $V_\infty^{K_\infty} \neq \{0\}$ only for the symmetric spaces $\mathrm{SO}(p+\infty)/\mathrm{SO}(p)\times\mathrm{SO}(\infty)$, $\mathrm{SU}(p+\infty)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(\infty))$ and $\mathrm{Sp}(p+\infty)/\mathrm{Sp}(p)\times \mathrm{Sp}(\infty)$ where $0 < p < \infty$. We then show that if $\{e_n\}$ is a sequence of K_n-invariant vectors in V_n

of norm 1 and $e = \lim_{n\to\infty} e_n \in V_\infty^{K_\infty}$, then the function $\psi_\infty(x) := \langle e, \pi_\infty(x)e\rangle$ is spherical in the sense of (1.3), and

$$\psi_\infty(x) = \lim_{n\to\infty} \psi_n(x)$$

where $\psi_n(x) = \langle e_n, \pi_n(x)e_n\rangle$. See Theorem 7.2.

Further discussion of the finite rank case is given in Section 8. Using result from [13] we show that in the case of finite rank all K_∞-spherical representations of G_∞ are given by the limit construction in Section 4. This in particular shows that the characterization of the spherical representations in terms of highest weights is still valid in the finite rank case.

2. Propagation of Symmetric Spaces

In this section we give a short overview of injective limits and propagation of compact symmetric spaces, as needed for our considerations on limits of spherical representations. We refer to [11] and [17] for details.

Let $M = G/K$ be a riemannian symmetric space of compact type. Thus G is a connected semisimple compact Lie group with an involution θ such that

$$(G^\theta)_o \subseteqq K \subseteqq G^\theta$$

where $G^\theta = \{x \in G \mid \theta(x) = x\}$ and the subscript $_o$ denotes the connected component containing the identity element. For simplicity we assume that M is simply connected.

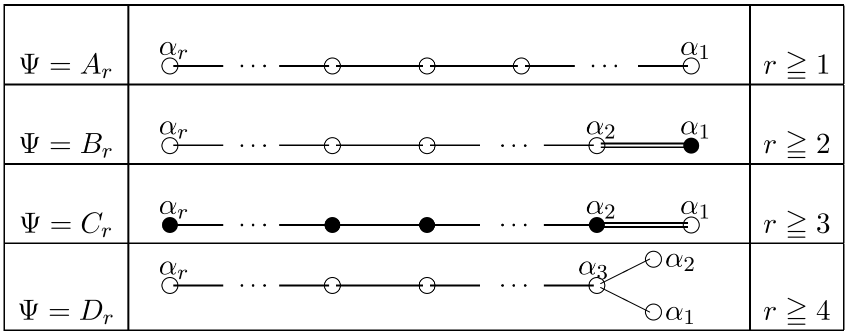
Denote the Lie algebra of G by $\mathfrak{g}$. By abuse of notation we write θ for the involution $d\theta : \mathfrak{g} \to \mathfrak{g}$. As usual $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ is the Lie algebra of K and $\mathfrak{s} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$. For $\alpha \in \mathfrak{a}_\mathbb{C}^*$ let

$$\mathfrak{g}_{\mathbb{C},\alpha} = \{X \in \mathfrak{g}_\mathbb{C} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}_\mathbb{C}\}.$$

If $\mathfrak{g}_{\mathbb{C},\alpha} \neq \{0\}$ then α is called a (restricted) root. Denote by $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subset i\mathfrak{a}^*$ the set of roots. Let $\Sigma_0 = \Sigma_0(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Sigma \mid 2\alpha \notin \Sigma\}$, the set of nonmultipliable roots. Then Σ_0 is a root system in the usual sense and the Weyl group corresponding to $\Sigma(\mathfrak{g}, \mathfrak{a})$ is the same as the Weyl group generated by the reflections s_α, $\alpha \in \Sigma_0$. Furthermore, M is irreducible as a riemannian symmetric space if and only if Σ_0 is irreducible as a root system.

Let $\Sigma^+ \subset \Sigma$ be a positive system and $\Sigma_0^+ = \Sigma^+ \cap \Sigma_0$. Then Σ_0^+ is a positive system in Σ_0. Denote by $\Psi = \{\alpha_1, \ldots, \alpha_r\}$, $r = \dim \mathfrak{a}$, the set of simple roots in Σ_0^+. Since we will be dealing with direct limits we may assume that Σ, and hence Σ_0, is one of the classical root systems. In order to facilitate considerations of direct limits, we number the simple roots in the following way:

(2.1)



The classical irreducible symmetric spaces are given by the following table. For the grassmannians we always assume that $p \leqq q$ and we let $n = p + q$. For $\alpha \in \Sigma$ we write $m_\alpha = \dim \mathfrak{g}_{\mathbb{C},\alpha}$, and for the simple roots we write $m_j = m_{\alpha_j}$. For the realization of each root system see [1], [3, Chapter 10] or [10]. In all these classical cases $m_{\alpha_j/2} = 0$ for $j > 1$. We will go into more detail in Section 3.

(2.2)

Irreducible compact riemannian symmetric $M = G/K$, G classical, K connected						
	G	K	Ψ	m_j $j > 1$	m_1	$m_{\alpha_1/2}$
1	$\mathrm{SU}(n) \times \mathrm{SU}(n)$	$\mathrm{diag}\,\mathrm{SU}(n)$	A_{n-1}	2	2	0
2	$\mathrm{Spin}(2n+1)\times$ $\mathrm{Spin}(2n+1)$	$\mathrm{diag}\,\mathrm{Spin}(2n+1)$	B_n	2	2	0
3	$\mathrm{Spin}(2n)\times$ $\mathrm{Spin}(2n)$	$\mathrm{diag}\,\mathrm{Spin}(2n)$	D_n	2	2	0
4	$\mathrm{Sp}(n) \times \mathrm{Sp}(n)$	$\mathrm{diag}\,\mathrm{Sp}(n)$	C_n	2	2	0
5	$\mathrm{SU}(n)$	$\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$	C_p	2	1	$2(q-p)$
6	$\mathrm{SU}(n)$	$\mathrm{SO}(n)$	A_{n-1}	1	1	0
7	$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$	A_{n-1}	4	4	0
8	$\mathrm{SO}(n)$	$\mathrm{SO}(p) \times \mathrm{SO}(q)$	B_p	1	$q-p$	0
9_1	$\mathrm{SO}(4n)$	$\mathrm{U}(2n)$	C_n	4	1	0
9_2	$\mathrm{SO}(2(2n+1))$	$\mathrm{U}(2n+1)$	C_n	4	1	4
10	$\mathrm{Sp}(n)$	$\mathrm{Sp}(p) \times \mathrm{Sp}(q)$	C_p	4	3	$4(q-p)$
11	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$	C_n	1	0	0

Cases (5), (8), and (10) are the grassmannians of p–planes in $\mathbb{F}^n$, $n = p + q$, where $\mathbb{F} = \mathbb{C}$, $\mathbb{R}$ or $\mathbb{H}$, respectively. In cases (5) and (10), $m_{\alpha_1/2} = (q-p)d$ and $m_{\alpha_1} = d - 1$ where $d = \dim_{\mathbb{R}} \mathbb{F}$. It is therefore more natural to view (8) as of type C_p with $m_{\alpha_1/2} = q - p$ and $m_{\alpha_1} = d - 1 = 0$.

We now assume that $\{M_k = G_k/K_k\}_{k \geqq 1}$ is a sequence of compact symmetric spaces such that $G_n \subsetneqq G_k$ and $K_n = G_n \cap K_k$ for $n \leqq k$. Then $M_n \subsetneqq M_k$. We write Σ_n, Σ_n^+, $\Sigma_{0,n}$, Ψ_n, etc. when we need to indicate dependence on the symmetric space M_n. We say that M_k *propagates* M_n if (i) $\mathfrak{a}_k = \mathfrak{a}_n$, or (ii) by choosing $\mathfrak{a}_n \subsetneqq \mathfrak{a}_k$ we obtain the Dynkin diagram in Table 2.1 for Ψ_k from that of Ψ_n by only adding simple roots at the left end. Then in particular Ψ_n and Ψ_k are of the same type.

In [10], [11] and [12] we used the set of indivisible roots instead of the set of nonmultipliable roots. Both definitions are equivalent.

When $\mathfrak{g}_k$ propagates $\mathfrak{g}_n$, and θ_k and θ_n are the corresponding involutions with $\theta_k|_{\mathfrak{g}_n} = \theta_n$, the corresponding eigenspace decompositions $\mathfrak{g}_k = \mathfrak{k}_k \oplus \mathfrak{s}_k$ and $\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{s}_n$ give us

$$\mathfrak{k}_n = \mathfrak{k}_k \cap \mathfrak{g}_n\,, \quad \text{and} \quad \mathfrak{s}_n = \mathfrak{g}_n \cap \mathfrak{s}_k \quad \text{for} \quad k \geqq n.$$

We recursively choose maximal commutative subspaces $\mathfrak{a}_k \subset \mathfrak{s}_k$ such that $\mathfrak{a}_n \subseteqq \mathfrak{a}_k$ for $k \geqq n$. We then have $\Sigma_n \subseteqq \Sigma_k|_{\mathfrak{a}_n} \setminus \{0\}$. We choose the positive ordering such that $\Sigma_n^+ \subseteqq \Sigma_k^+|_{\mathfrak{a}_n} \setminus \{0\}$.

Note that by moving along each row in Table 2.2 we have a propagation of symmetric spaces. In all cases except (5), (8) and (10) the multiplicities remain constant, in fact less or equal to 4.

We set

$$G_\infty = \varinjlim G_n \quad K_\infty = \varinjlim K_n\,, \text{ and } M_\infty = \varinjlim M_n = G_\infty/K_\infty\,.$$

In this paper we consider the question of whether the inductive limit of K_n–spherical representations of G_n is K_∞–spherical. For that we need to recall the construction of inductive limits of spherical representations, the theory of highest weights of spherical representations and the Harish–Chandra c–function of the noncompact dual of G_n.

3. Spherical Representations of Compact Groups

In this section we give a short overview of spherical representations, their highest weights, and connections to propagation of symmetric spaces. Most of the material can be found in [10], [11], [17], [16] and [15]. The notation will be as in Section 2, and G or G_n will always stand for a connected compact group. If $k \geqq n$ then we assume that $G_n \subseteqq G_k$ and that M_k propagates M_n. We also assume that each of the symmetric spaces M_n is simply connected. We denote by r_k and r_n the respective real ranks of M_k and M_n. As always we fix compatible K_k– and K_n–invariant inner products on $\mathfrak{s}_k$, respectively $\mathfrak{s}_n$.

For a representation (π, V) of G let $V^K = \{u \in V \mid (\forall k \in K)\ \pi(k)u = u\}$. If (π, V) is irreducible, then we say that (π, V), or simply π, is K–spherical, or just spherical, if $V^K \neq \{0\}$. It is well known that π is spherical if and only if $\dim V^K = 1$. Furthermore, in that case the highest weight of π is contained in $i\mathfrak{a}^*$. Let

$$\Lambda^+ = \Lambda^+(G,K) = \left\{ \mu \in i\mathfrak{a}^* \;\middle|\; \tfrac{\langle \mu,\alpha\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}^+ \quad \text{for all} \quad \alpha \in \Sigma^+ \right\}. \tag{3.1}$$

Theorem 3.2 (Cartan–Helgason). *Let (π, V) be an irreducible representation of G, and μ its highest weight. Then the following are equivalent.*

1. *(π, V) is spherical.*
2. *$\mu \in i\mathfrak{a}^*$ and $\mu \in \Lambda^+$.*
3. *The multiplicity of (π, V) in $L^2(M)$ is 1.*
4. *π is a subrepresentation of the representation of G in $L^2(M)$.*

See [4, Theorem 4.1, p. 535] for the proof.

From now on, if $\mu \in \Lambda^+$ then π_μ denotes the irreducible spherical representation with highest weight μ. Define linear functionals $\xi_j \in i\mathfrak{a}^*$ by

$$\frac{\langle \xi_i, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle} = \delta_{i,j} \quad \text{for} \quad 1 \leqq j \leqq r \ . \tag{3.3}$$

Then $\xi_1, \ldots, \xi_r \in \Lambda^+$ and

$$\Lambda^+ = \mathbb{Z}^+\xi_1 + \ldots + \mathbb{Z}^+\xi_r = \left\{ \sum_{j=1}^{r} n_j \xi_j \;\middle|\; n_j \in \mathbb{Z}^+ \right\} .$$

The weights ξ_j are called the *class 1 fundamental weights for* $(\mathfrak{g}, \mathfrak{k})$. Set $\Xi = \{\xi_1, \ldots, \xi_r\}$. We always have

$$\rho = \sum_{j=1}^{r} \rho_j \xi_j \quad \text{with} \quad \rho_j = \frac{1}{2}\left(m_{\alpha_j} + \frac{m_{\alpha_j/2}}{2}\right) .$$

We write $a = \frac{1}{2}\left(m_{\alpha_1} + \frac{m_{\alpha_1/2}}{2}\right)$. Using $m_{\alpha_j} = m_{\alpha_i}$ for $i, j \geqq 2$ we set $b = \frac{1}{2} m_{\alpha_j}$, $j \geqq 2$. Then

$$\rho = a\xi_1 + b\sum_{j=2}^{r} \xi_j \, . \tag{3.4}$$

We will need a particular formulation for each classical root system. We identify $\mathfrak{a}$ with $\mathbb{R}^r$ so that, as usual, $\mathfrak{a} = \{(x_{r+1}, \ldots, x_1) \mid x_1 + \ldots + x_{r+1} = 0\}$ if $\Psi = A_r$ and otherwise $\mathfrak{a} = \mathbb{R}^r$. Set $f_1 = (0, \ldots, 0, 1)$, $f_2 = (0, \ldots, 0, 1, 0)$, $\ldots$, $f_n = (1, 0, \ldots, 0)$ where $n = r + 1$ for A_r and otherwise $n = r$.

For $\Psi = A_r$ we have $\Sigma_0^+ = \{f_j - f_i \mid 1 \leqq i < j \leqq n\}$, and $\alpha_j = f_{j+1} - f_j$, $j = 1, \ldots, r$. We have

$$\xi_j = 2\sum_{i=j+1}^{r+1} f_i \, .$$

Thus

$$\Lambda^+ \simeq \{(m_r, m_{r-1}, \ldots, m_1, 0) \in (2\mathbb{Z}^+)^{r+1} \mid m_i \leqq m_j \text{ if } i < j\} \, .$$

The multiplicities are constant, equal to $m = 1$, 2 or 4. Hence $a = b = 1/2, 1$, or 2 and we have

$$\rho = a\sum_{j=1}^{r} \xi_j = m(r, r-1, \ldots, 1, 0) = 2a\sum_{j=1}^{r+1} (j-1) f_j \, . \tag{3.5}$$

If Ψ is of type B_r then we have $\Sigma_0^+ = \{f_j \mid j = 1, \ldots, r\} \cup \{f_j \pm f_i \mid 1 \leqq i < j \leqq r\}$ and $\Psi = \{\alpha_1 = f_1\} \cup \{\alpha_i = f_i - f_{i-1} \mid i = 2, \ldots, r\}$. Thus

$$\xi_1 = \sum_{j=1}^{r} f_j \text{ and } \xi_j = 2\sum_{i=j}^{r} f_i \, , \; j > 1 \, .$$

In particular

$$\Lambda^+ \simeq \{(m_r, \dots, m_1) \in (\mathbb{Z}^+)^r \mid m_i \leqq m_j \text{ and } m_j - m_i \text{ even for } i < j\}\,.$$

Finally we have

$$\rho = \sum_{j=1}^{n} \xi_j = (2r-1, 2r-3, \dots, 3, 1) = \sum_{j=1}^{r} (2j-1) f_j \tag{3.6}$$

in case (2). Case (8), which is the other possibility for Ψ of type B, will be considered in the discussion of C_r, as explained just after Table 2.2.

If Ψ is of type C_r then we have $\Sigma_0^+ = \{2f_j \mid j = 1, \dots, r\} \cup \{f_j \pm f_i \mid 1 \leqq i < j \leqq r\}$ and $\Psi = \{\alpha_1 = 2f_1\} \cup \{\alpha_j = f_j - f_{j-1} \mid j = 2, \dots, r\}$. Thus

$$\xi_j = 2 \sum_{i=j}^{r} f_i$$

and

$$\rho = 2a \sum_{j=1}^{r} f_j \; + \; 2b \sum_{\nu=2}^{r} \sum_{j=\nu}^{r} f_j = 2 \sum_{j=1}^{r} (a + b(j-1)) f_j\,. \tag{3.7}$$

There is just one case where Ψ is of type D_r. There $a = b = 1$. In that case we have $\alpha_1 = f_1 + f_2$ and $\alpha_j = f_j - f_{j-1}$ for $j \geqq 2$. Thus

$$\xi_1 = \sum_{i=1}^{r} f_i\,,\ \xi_2 = -f_1 + \sum_{j=2}^{r} f_j\,,\ \text{and } \xi_j = 2 \sum_{i=j}^{r} f_i \ \text{ for } j \geqq 3\,.$$

That gives us

$$\rho = 2 \sum_{j=2}^{r} (j-1) f_j\,. \tag{3.8}$$

Fix a $\mu \in \Lambda^+$ and let (π_μ, V_μ) be the corresponding spherical representation of G. Fix a highest weight vector $u_\mu \in V_\mu$ and a K–fixed vector e_μ. We assume that $\|u_\mu\| = \|e_\mu\| = 1$. For the following it is important to evaluate the inner product $\langle u_\mu, e_\mu \rangle$ in a systematic way so that we can control it as we consider inductive limits of spherical representations in the next section. The following is well known, but we include the proof for completeness.

First of all we always have $\langle u_\mu, e_\mu \rangle \neq 0$. We choose u_μ and e_μ so that $\langle u_\mu, e_\mu \rangle > 0$.

Let $\mathfrak{g}' = \mathfrak{k} \oplus i\mathfrak{s}$. As G is compact it is a linear group, thus contained in a complex linear group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let G' be the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}'$. Note that the holomorphic extension of θ to $\mathfrak{g}_{\mathbb{C}}$ restricted to $\mathfrak{g}'$ defines a Cartan involution on $\mathfrak{g}'$. We also denote this involution and the corresponding Cartan involution on G' by θ. Let $\overline{N} = \theta(N)$. Then G' has a Iwasawa decomposition (recall that we are assuming G simply connected, in particular K is connected)

$$G' = KA'N \tag{3.9}$$

where $A' = \exp(i\mathfrak{a})$ and the Lie algebra of N is $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}'_\alpha$.

For $x \in G'$ write $x = k(x)a(x)n(x)$ according to the Iwasawa decomposition (3.9). We normalize the Haar measure on $\overline{N}$ such that

$$\int_{\overline{N}} a(\bar{n})^{-2\rho}\, d\bar{n} = 1\,,$$

Then the integral

$$\mathbf{c}(\lambda) = \int_{\overline{N}} a(\bar{n})^{-\lambda-\rho}\, d\bar{n}$$

converges for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ such that $\mathrm{Re}\,\langle\lambda,\alpha\rangle > 0$ for all $\alpha \in \Sigma^+$. The function $\mathbf{c}(\lambda)$ is the Harish–Chandra c–function. It has a meromorphic continuation to all of $\mathfrak{a}^*_{\mathbb{C}}$ and is given by

$$\mathbf{c}(\lambda) = \frac{{}'c(\lambda)}{{}'c(\rho)}$$

where ${}'c(\lambda)$ is explicitly given by the Gindikin–Karpelevich product formula. In terms of Σ_0^+, we have

$$ {}'c(\lambda) = \prod_{\alpha\in\Sigma_0^+} {}'c_\alpha(\lambda_\alpha) \tag{3.10}$$

where

$$ {}'c_\alpha(\lambda_\alpha) = \frac{2^{-2\lambda_\alpha}\ \Gamma(2\lambda_\alpha)}{\Gamma\left(\lambda_\alpha + \frac{m_{\alpha/2}}{4} + \frac{1}{2}\right)\Gamma\left(\lambda_\alpha + \frac{m_{\alpha/2}}{4} + \frac{m_\alpha}{2}\right)}\,, \quad \lambda_\alpha = \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \tag{3.11}$$

where Γ is the Euler Γ–function $\Gamma(x) := \int_0^\infty e^{-t}t^{x-1}dt$. Formula (3.11) looks slightly different from the usual formula for the c–function as found for instance in [4], Ch. IV, Theorem 6.4, where it is written in terms of positive indivisible roots ($\alpha \in \Sigma^+$ with $\alpha/2 \notin \Sigma^+$) rather than in terms of positive nonmultipliable roots. The formula (3.10) was used in [9]. The equivalence of the two formulas follows from the doubling formula $\sqrt{\pi}\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})$ for the Gamma function.

Let $M = Z_K(\mathfrak{a})$. The following can be found in any standard reference on symmetric spaces.

Lemma 3.12. *Let $f \in L^1(K/M)$. Then*

$$\int_{K/M} f(kM)\, d(kM) = \int_{\overline{N}} f(k(\bar{n})M)\, a(\bar{n})^{-2\rho}\, d\bar{n}\,.$$

Clearly the vector $\int_K \pi_\mu(k)u_\mu\, dk$ is K–invariant, and by taking the inner product with e_μ one gets $\int_K \pi_\mu(k)u_\mu\, dk = \langle u_\mu, e_\mu\rangle e_\mu$.

Lemma 3.13. $\left\langle \int_K \pi_\mu(k)u_\mu dk, u_\mu \right\rangle = \int_K \langle \pi_\mu(k)u_\mu, u_\mu\rangle\, dk = \mathbf{c}(\mu+\rho)$.

Proof. The proof is a simple calculation using Lemma 3.12. To simplify the notation we write $u = u_\mu$ and π for π_μ. The representation π extends to a holomorphic representation of $G_{\mathbb{C}}$. Hence

$\pi(x)$ is well defined for $x \in G'$. We note that $\pi(mn)u = u$ for all $n \in N$ and $m \in M$ (see [4], Theorem 4.1, p. 535). In particular $\langle \pi(\bar{n})u, u\rangle = \langle u, \pi(\theta(n)^{-1})u\rangle = \langle u, u\rangle = 1$.

$$\begin{aligned}
\int_{K/M} \langle \pi(k)u, u\rangle d(kM) &= \int_{\overline{N}} \langle \pi(k(\overline{n}))u, u\rangle a(\overline{n})^{-2\rho}\, d\overline{n} \\
&= \int_{\overline{N}} \langle \pi(\overline{n}a(\overline{n})^{-1}n')u, u\rangle\, a(\overline{n})^{-2\rho}\, d\overline{n} \qquad \text{where } n' \in N \\
&= \int_{\overline{N}} \langle \pi(\overline{n})u, u\rangle\, a(\overline{n})^{-\mu-2\rho}\, d\overline{n} \\
&= \int_{\overline{N}} a(\bar{n})^{-\mu-2\rho}\, d\bar{n} \\
&= \mathbf{c}(\mu+\rho)\,.
\end{aligned}$$

That proves the Lemma. □

Theorem 3.14. *Let u_μ and e_μ be as above. Then $\langle u_\mu, e_\mu\rangle = \sqrt{\mathbf{c}(\mu+\rho)}$. In particular*

$$\int_K \pi_\mu(k)u_\mu\, dk = \sqrt{\mathbf{c}(\mu+\rho)}\, e_\mu\,.$$

Proof. We use the same notation as in Lemma 3.13. By that lemma, we see that

$$\begin{aligned}
\mathbf{c}(\mu+\rho) &= \left\langle \int_K \pi_\mu(k)u_\mu dk, u_\mu \right\rangle \\
&= \langle \langle u_\mu, e_\mu\rangle e_\mu, u_\mu\rangle \\
&= \langle u_\mu, e_\mu\rangle^2
\end{aligned}$$

Thus we see that $\langle u_\mu, e_\mu\rangle = \sqrt{\mathbf{c}(\mu+\rho)}$. The rest of the result follows when we recall that $\int_K \pi_\mu(k)u_\mu\, dk = \langle u_\mu, e_\mu\rangle e_\mu$. □

Theorem 3.15. *For $\alpha \in \Sigma_0^+$ let $x_\alpha := \frac{1}{4}\left(m_{\alpha/2}+2\right)$ and $y_\alpha = \frac{1}{4}\left(m_{\alpha/2}+2m_\alpha\right)$. Let $\mu \in \Lambda^+$. Then $\mu_\alpha \in \mathbb{Z}^+$ for all $\alpha \in \Sigma_0^+$ and*

$$\mathbf{c}(\mu+\rho) = \prod_{\alpha\in\Sigma_0^+} \left(\left(1+\frac{x_\alpha}{\rho_\alpha}\right)\left(1+\frac{y_\alpha}{\rho_\alpha}\right)\right)^{-\mu_\alpha} \cdot \prod_{j=0}^{\mu_\alpha-1} \frac{\left(1+\frac{j}{2\rho_\alpha}\right)\left(1+\frac{\mu_\alpha+j}{2\rho_\alpha}\right)}{\left(1+\frac{j}{x_\alpha+\rho_\alpha}\right)\left(1+\frac{j}{y_\alpha+\rho_\alpha}\right)} \tag{3.16}$$

where the product $\displaystyle\prod_{j=0}^{\mu_\alpha-1}$ is understood to be 1 if $\mu_\alpha = 0$.

Proof. By (3.10) and (3.11) we can write $\mathbf{c}(\mu+\rho) = \prod_{\alpha\in\Sigma_0^+} \mathbf{c}_\alpha(\mu_\alpha+\rho_\alpha)$ with

$$\mathbf{c}_\alpha(\mu_\alpha+\rho_\alpha) = \frac{2^{-2\mu_\alpha}\Gamma(2(\mu_\alpha+\rho_\alpha))}{\Gamma(2\rho_\alpha)} \frac{\Gamma(\rho_\alpha+x_\alpha)}{\Gamma(\mu_\alpha+\rho_\alpha+x_\alpha)} \frac{\Gamma(\rho_\alpha+y_\alpha)}{\Gamma(\mu_\alpha+\rho_\alpha+y_\alpha)}\,.$$

Now, using that $\mu_\alpha \in \mathbb{Z}^+$ and $\Gamma(x+1) = x\Gamma(x)$, we get for $\mu_\alpha \neq 0$:

$$\begin{aligned}
\Gamma(2(\mu_\alpha + \rho_\alpha)) &= \left(\prod_{j=1}^{2\mu_\alpha}(2(\mu_\alpha + \rho_\alpha) - j)\right)\Gamma(2\rho_\alpha) \\
&= \left(\prod_{j=0}^{2\mu_\alpha - 1}(2\rho_\alpha + j)\right)\Gamma(2\rho_\alpha) \\
&= 2^{2\mu_\alpha}\rho_\alpha^{2\mu_\alpha}\left(\prod_{j=0}^{\mu_\alpha - 1}\left(1 + \frac{j}{2\rho_\alpha}\right)\left(1 + \frac{\mu_\alpha + j}{2\rho_\alpha}\right)\right)\Gamma(2\rho_\alpha)\,.
\end{aligned}$$

Similarly

$$\frac{\Gamma(\rho_\alpha + x_\alpha)}{\Gamma(\mu_\alpha + \rho_\alpha + x_\alpha)}\frac{\Gamma(\rho_\alpha + y_\alpha)}{\Gamma(\mu_\alpha + \rho_\alpha + y_\alpha)} = \prod_{j=0}^{\mu_\alpha - 1}\frac{1}{(\rho_\alpha + x_\alpha + j)(\rho_\alpha + y_\alpha + j)}$$

and the claim follows. □

4. Inductive Limits of Spherical Representations

In this section we recall the construction of inductive limits of spherical representations [17, Section 3] and [11]. We always assume that $k \geqq n$ and that we have a sequence $\{M_k = G_k/K_k\}$ such that M_k propagates M_n. The index k (respectively n) will indicate objects related to G_k (respectively G_n). As in [15] or [17], our description of the root system and the fundamental weights gives

Lemma 4.1. *Assume that M_k propagates M_n. Let*

$$\Psi_n = \{\alpha_{n,1}, \dots \alpha_{n,r_n}\} \text{ and } \Xi_n = \{\xi_{n,1}, \dots, \xi_{n,r_n}\}$$

and similarly for M_k. Assume that $j \leqq r_n$. Then

1. *$\alpha_{k,j}$ is the unique element of Ψ_k whose restriction to $\mathfrak{a}_n$ is $\alpha_{n,j}$.*
2. *If $\mu_n = \sum_{j=1}^{r_n} k_j\xi_{n,j} \in \Lambda_n^+$, then $\mu_k := \sum_{j=1}^{r_n} k_j\xi_{k,j} \in \Lambda_k^+$ and $\mu_k|_{\mathfrak{a}_n} = \mu_n$.*

For $I = (k_1, \dots, k_{r_n}) \in (\mathbb{Z}^+)^{r_n}$ define $\mu_I := \mu(I) = k_1\xi_{n,1} + \ldots + k_{r_n}\xi_{n,r_n}$. Lemma 4.1 allows us to form direct system of representations, as follows. For $\ell \in \mathbb{N}$ denote by $0_\ell = (0, \dots, 0)$ the zero vector in $\mathbb{R}^\ell$. For $I_n = (k_1, \dots, k_{r_n}) \in (\mathbb{Z}^+)^{r_n}$ let

(4.2)

- $\mu_{I,n} = \mu(I_n) = \sum_{j=1}^{r_n} k_j\xi_{n,j} \in \Lambda_n^+$;
- $\pi_{I,n} = \pi_{\mu(I_n)}$ the corresponding spherical representation;
- $V_{I,n} = V_{\mu(I_n)}$ a fixed Hilbert space for the representation $\pi_{I,n}$;
- $u_{I,n} = u_{\mu(I_n)}$ a highest weight unit vector in $V_{I,n}$, $\|u_{I,n}\| = 1$;
- $e_{I,n} = e_{\mu(I_n)}$ a K_n–fixed unit vector in $V_{I,n}$ such that $\langle u_{I,n}, e_{I,n}\rangle > 0$

Later, I_n will be fixed and we will write π_n, V_n, μ_n, u_n, and e_n without further comments.

Theorem 4.3. *Assume that M_k propagates M_n. Let $(\pi_{I,n}, V_{I,n})$ be an irreducible spherical representation of G_n with highest weight $\mu_{I,n} \in \Lambda_n^+$. Let $I_k = (I_n, 0_{r_k - r_n})$. Then the following hold.*

1. *The G_n–submodule of $V_{I,k}$ generated by $u_{I,k}$ is irreducible and isomorphic to $V_{I,n}$.*
2. *The multiplicity of $\pi_{I,n}$ in $\pi_{I,k}|_{G_n}$ is 1.*

Remark 4.4. From this point on, when $n \leqq k$ we will always assume that the Hilbert space $V_{I,n}$ is realized inside $V_{I,k}$ as the span of $\pi_{I,k}(G_n)u_{I,k}$, in other words by identification of highest weight unit vectors. Thus we can then assume that $u_{I,n} = u_{I,k}$. On the other hand we almost never have $e_{I,n} = e_{I,k}$ under this inclusion. But we can always assume that $e_{I,k} = q(k,n)\, e_{I,n} + e_{k,n}^{\perp}$ where $q(k,n) = \langle e_{I,k}, e_{I,n}\rangle > 0$ and $\langle e_{I,n}, e_{k,n}^{\perp}\rangle = 0$. One of our aims is to evaluate $\langle e_{I,k}, e_{I,n}\rangle$ in terms of c–functions. $\diamondsuit$

Theorem 4.3 allows us to define a inductive limit of spherical representation starting by a given spherical representation $(\pi_{I,n}, V_{I,n})$ of G_n. We have isometric embeddings $V_n = V_{I,n} \hookrightarrow V_{n+1} = V_{I,n+1}$ defined by the map $u_n = u_{I,n} \mapsto u_{n+1} = u_{I,n+1}$. As $u_{I,n}$ is independent of n we simply write u for the fixed highest weight vector. Then the algebraic inductive limit $\varinjlim V_\infty$ is a pre–Hilbert space with inner product $\langle v, w\rangle = \langle v, w\rangle_{V_k}$ if $v, w \in V_k$. This inner product is well defined as the embeddings $V_n \hookrightarrow V_k$ are isometric. We denote by $\mu_\infty = \varinjlim \mu_{I,n} \in i\mathfrak{a}_\infty^*$ and $V_{\infty,\mu_\infty} = V_\infty$ the Hilbert space completion of $\varinjlim V_n$. Notice that $u \in V_\infty$.

Our main theorem in this article is the following theorem:

Theorem 4.5. *Let the notation be as above and assume that $\mu \neq 0$. Then $V_\infty^{K_\infty} \neq \{0\}$ if and only if the ranks of the compact riemannian symmetric spaces M_n are bounded. Thus $V_\infty^{K_\infty} \neq \{0\}$ only for the symmetric spaces* $\mathrm{SO}(p+\infty)/\mathrm{SO}(p)\times\mathrm{SO}(\infty)$, $\mathrm{SU}(p+\infty)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(\infty))$ *and* $\mathrm{Sp}(p+\infty)/\mathrm{Sp}(p)\times\mathrm{Sp}(\infty)$ *where $0 < p < \infty$.*

Here is our strategy. First, if $V_\infty^{K_\infty} \neq \{0\}$ let $e_\infty \in V_\infty^{K_\infty}$ be a unit vector. Then consider the projection $\mathrm{proj}_{\infty,n}(e_\infty) \in V_n^{K_n} \setminus \{0\}$. Let $e_n = \mathrm{proj}_{\infty,n}(e_\infty)/\|\mathrm{proj}_{\infty,n}(e_\infty)\|$. Then $\{e_n\}$ is a Cauchy sequence such that e_n is a unit vector in $V_n^{K_n}$ and $\{e_n\} \to e_\infty$. On the other hand if $\{e_n\}$ is a Cauchy sequence in V_∞ such that $e_n \in V_n^{K_n}$ and $\|e_n\| = 1$, then $e_\infty = \lim e_n$ is a nonzero element of $V_\infty^{K_\infty}$. Thus $V_\infty^{K_\infty} \neq \{0\}$ if and only if we can find a Cauchy sequence $\{e_n\}$ such that $e_n \in V_n^{K_n}$ is a unit vector.

Recursively choose K_{n+1}–fixed unit vectors e_{n+1} so that the orthogonal projection of V_{n+1} onto V_n sends e_{n+1} to a positive real multiple of e_n as mentioned before. Then $\mathrm{proj}_{m,n}(e_m) = q(m,n)e_n$ for $m \geqq n$ where the $q(m,n)$ are real with $0 < q(m,n) \leqq 1$. Since $\mathrm{proj}_{m,\ell} = \mathrm{proj}_{n,\ell}\cdot\mathrm{proj}_{m,n}$ we have $q(n,\ell)q(m,n) = q(m,\ell)$. Furthermore, for a fixed n the function $m \mapsto q(m,n)$ is decreasing. Also, choosing e_1 such that $\langle u, e_1\rangle$ is positive real we have $\langle u, e_n\rangle$ positive real for all n.

Theorem 4.6. *Let $m \geqq n$. Then $\langle e_m, e_n\rangle = \sqrt{\mathbf{c}_m(\mu_m+\rho_m)/\mathbf{c}_n(\mu_n+\rho_n)}$.*

Proof. To simplify the notation we write μ for both μ_m and μ_n and $\mathbf{c}_n(\mu+\rho)$ for $\mathbf{c}_n(\mu_n+\rho_n)$. From Lemma 3.14 we have $e_m = \mathbf{c}_m(\mu+\rho)^{-1/2}\int_{K_m}\pi_m(k)u\,dk$ and similarly for e_n. So

$$\begin{aligned}
\langle e_m, e_n\rangle &= (\mathbf{c}_m(\mu+\rho)\mathbf{c}_n(\mu+\rho))^{-1/2}\int_{k\in K_n}\int_{h\in K_m}\langle \pi_m(h)u, \pi_n(k)u\rangle\,dk\,dh\\
&= (\mathbf{c}_m(\mu+\rho)\mathbf{c}_n(\mu+\rho))^{-1/2}\int_{k\in K_n}\int_{h\in K_m}\langle \pi_m(k^{-1}h)u, u\rangle\,dh\,dk\\
&= (\mathbf{c}_m(\mu+\rho)\mathbf{c}_n(\mu+\rho))^{-1/2}\int_{h\in K_m}\langle \pi_m(h)u, u\rangle\,dh \qquad \text{as } K_n \subsetneqq K_m\\
&= (\mathbf{c}_m(\mu+\rho)\mathbf{c}_n(\mu+\rho))^{-1/2}\mathbf{c}_m(\mu+\rho)\\
&= \sqrt{\mathbf{c}_m(\mu+\rho)/\mathbf{c}_n(\mu+\rho)}
\end{aligned}$$

as asserted. □

Theorem 4.7. *The limit* $\lim_{m\to\infty}\mathbf{c}(\mu_m+\rho_m)$ *exists and is non–negative. Let the sequence* $\{e_n\}_n$ *be as before. Then* $\{e_n\}$ *converges to a nonzero element* $e \in V_\infty^{K_\infty}$ *if and only if* $\lim \mathbf{c}_m(\mu_m+\rho_m) > 0$.

Proof. We start by observing

$$\|e_m - e_n\|^2 = \|e_n\|^2 - 2\langle e_m, e_n\rangle + \|e_m\|^2 = 2(1 - \langle e_m, e_n\rangle)\,.$$

Hence $\{e_n\}$ is a Cauchy sequence if and only if

$$\lim_{m,n\to\infty}\langle e_m, e_n\rangle = \lim_{m,n\to\infty}\sqrt{\mathbf{c}_m(\mu_m+\rho_m)/\mathbf{c}_n(\mu_n+\rho_n)} = 1\,.$$

For fixed n the sequence $\langle e_m, e_n\rangle \geqq 0$ is decreasing and bounded below by zero. Hence $\ell_n := \lim_m\langle e_m, e_n\rangle$ exists. This implies that the limit $\lim_m \mathbf{c}(\mu_m+\rho_m)$ exists and is non–negative.

The sequence $0 \leqq \ell_n \leqq 1$ is either zero or increasing and hence $\lim \ell_n$ exists. It follows that $\lim\limits_{m,n\to\infty}\sqrt{\mathbf{c}_m(\mu_m+\rho_m)/\mathbf{c}_n(\mu_n+\rho_n)}$ exists (and thus has to be equal to 1) if and only if $\lim_{m\to\infty}\mathbf{c}_m(\mu_m+\rho_m) > 0$. □

5. The Finite Rank Cases

In this section we prove Theorem 4.5 for the finite rank cases, i.e. the cases where $M_n = \mathrm{SO}(n)/\mathrm{SO}(p)\times\mathrm{SO}(q)$, $\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$ or $\mathrm{Sp}(n)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$ with p fixed and $n = p+q$. We may assume $q \geqq p$, so all the M_n have the same finite rank p. The cardinality of Σ_0^+ is constant.

We use the notation from the previous section. Recall $d = \dim_{\mathbb{R}}\mathbb{F}$. View the real grassmannian $\mathrm{SO}(n)/\mathrm{SO}(p)\times\mathrm{SO}(q)$ as of type C_p with $m_{\alpha_1} = 0$ as explained after Table 2.2.

Note that $q \to \infty$ with n. Furthermore, the highest weight $\mu = \sum_{\nu=1}^{p} k_\nu\xi_\nu$ is independent of n. Now, in view of Theorem 3.15, it suffices to prove that

$$\frac{x_\alpha}{\rho_\alpha} = \frac{1}{4}\left(\frac{2}{\rho_\alpha} + \frac{m_{\alpha/2}}{\rho_\alpha}\right) \qquad \text{and} \qquad \frac{y_\alpha}{\rho_\alpha} = \frac{1}{4}\left(2\frac{m_\alpha}{\rho_\alpha} + \frac{m_{\alpha/2}}{\rho_\alpha}\right)$$

are bounded for all α.

For that we only need to consider where α is in the Weyl group orbit of α_1, because in all other cases m_α and $m_{\alpha/2}$ are bounded (in fact $\leqq 4$). We have

$$m_{\alpha_1} = d-1\,, \quad m_{\alpha_1/2} = d(q-p) \quad \text{and} \quad \rho_{\alpha_1} = \tfrac{1}{2}\left(d-1+\tfrac{(q-p)d}{2}\right).$$

Thus $\frac{m_{\alpha_1/2}}{\rho_{\alpha_1}}$ is bounded. That completes the proof of Theorem 4.5 for the finite rank cases.

6. The Infinite Rank Cases.

In this section we prove Theorem 4.5 for the infinite rank cases, i.e., the cases where $\operatorname{rank} M_n$ is unbounded. We may pass to a subsequence of $\{M_n\}$, and then of $\{n\}$, and assume that $\operatorname{rank} M_n = n$. Now we start the proof by reducing it to the case where $\mu_\alpha = 1$ in (3.16).

Lemma 6.1. *Assume that $\mu_m = \sum_{j=1}^n k_j \xi_{m,j}$ with $k_n > 0$. Then*

$$\mathbf{c}_m(\mu_m+\rho_m) \leq \mathbf{c}_m(\xi_{m,n}+\rho_m)\,.$$

Proof. By [4, Corollary 6.6, Ch IV] we have $\mu_{m,n}(\log(a(\bar{n}))) \leqq 0$, so $\mathbf{c}_m(\mu_m+\rho_m)$ is a decreasing function of μ_m. □

We will also need the following well known and simple fact:

Lemma 6.2. *Assume that $\epsilon, \delta > 0$. Let $a_j \geqq \epsilon$ and $0 \leqq x_j \leq \delta$. Then*

$$\lim_{N\to\infty} \prod_{j=L}^{N} \left(1+\tfrac{a_j}{x_j+j}\right)^{-1} = 0\,.$$

Proof. If $x > 0$ is small enough then $1+x \leqq e^{x/2}$. Hence

$$\prod_{j=L}^{N} \left(1+\tfrac{a_j}{x_j+j}\right) \geqq \exp\left(\epsilon \sum\nolimits_{j=L}^{N} \tfrac{1}{\delta+j}\right) \to \infty \text{ as } N\to\infty$$

and the claim follows. □

The idea of the proof of Theorem 4.5, for the unbounded rank cases, is to find a sequence of roots α_n such that $\xi_{n,k,\alpha_n} = 1$ and ρ_{n,α_n} is affine linear in n. Then, if $\alpha_n/2$ is not a root, $x_{\alpha_n} = 1/2$ and the expression for $\mathbf{c}_{\alpha_n}(\xi_{n,k,\alpha_n}+\rho_{n,\alpha_n})$ in (3.16) reduces to

$$\mathbf{c}_{\alpha_n}(\xi_{n,k,\alpha_n}+\rho_{n,\alpha_n}) = \left(1+\frac{y_{\alpha_n}}{\rho_{n,\alpha_n}}\right)^{-1}.$$

It will then follow from Lemma 6.2 that

$$\lim_{N\to\infty} \prod_{n=k}^{N} \mathbf{c}_{\alpha_n}(\xi_{n,k,\alpha_n}+\rho_{n,\alpha_n}) = 0\,.$$

That will finish the proof because $\mathbf{c}_{n,\alpha}(\xi_{n,k,\alpha}+\rho_{n,\alpha}) \leqq 1$ for all n and all positive roots.

In the case $\Psi = A_n$ we let $\alpha_n = f_{n+1} - f_1$. Then $\rho_{n,\alpha_n} = tn$, $t = 1/2, 1$ or 2, $\xi_{n,k,\alpha_n} = 1$, $x_{\alpha_n} = 1/2$, and $y_{\alpha_n} = t$, and the claim follows by the argument indicated above.

If $\Psi = B_n$ we take $\alpha_n = f_n - f_1$ when $k \neq 1$ and $\alpha_n = f_n$ when $k = 1$. Then $x_{\alpha_n} = 1/2$, $\xi_{n,k,\alpha_n} = 1$, ρ_{n,α_n} is affine linear in n, and the claim follows as in the A_n case using the argument indicated above.

When $\Psi = C_n$ we take $\alpha_n = f_n + f_1$. Then both the multiplicities and the ρ_{n,α_n} increase affinely in n, to the claim follows as above.

In the one D_n case we take $\alpha_n = f_n + f_2$ for n large and the same argument goes through. This completes the proof of Theorem 4.5. $\square$

7. The Connection to Spherical Functions on G_∞

Finally, we discuss the connection with the theory of K_∞–spherical functions on G_∞ as developed by Olshanskii, Faraut and coworkers. See [2] for references. Those authors define a nonzero continuous function $\varphi : G_\infty \to \mathbb{C}$ to be *spherical* if for all $x, y \in G_\infty$

$$\lim_{n\to\infty} \int_{K_n} \varphi(xky)\, dk = \varphi(x)\varphi(y)\,. \tag{7.1}$$

By taking x, respectively y to be the identity it is clear that a spherical function is K_∞–biinvariant and takes the value 1 at the identity.

Theorem 7.2. *Assume that rank G_∞/K_∞ is finite and $V_\infty = \varinjlim V_{n,\mu_n}$. Let $\{e_n\}_n$ be a Cauchy sequence in V_∞ such that for all n $\|e_n\| = 1$, $e_n \in V_{n,\mu_n}^{K_n}$ and $e_n \to e_\infty \in V_\infty^{K_\infty}$. Then*

$$\varphi_{\mu_\infty}(x) := \langle e_\infty, \pi_{\infty,\mu_\infty}(x) e_n\rangle = \lim_{n\to\infty} \langle e_n, \pi_{n,\mu_n}(x) e_n\rangle$$

is a positive definite K_∞–spherical function on G_∞ in the sense of (7.1).

Proof. Write $e_\infty = e_n + e_n^\perp$. Let $x, y \in G_{j_o}$. Then, for $j \geqq j_o$,

$$\varphi_{\mu_\infty}(x) = \langle e_j, \pi_j(x) e_j\rangle = \varphi_{\mu_j}(x) + \langle e_j^\perp, \pi_j(x) e_j^\perp\rangle$$

because V_j and $V_j^\perp$ are K_j–invariant. Thus

$$|\langle e_j^\perp, \pi_j(x) e_j^\perp\rangle| \leqq \|e_j^\perp\|^2 \to 0\,.$$

Hence $\varphi_{\mu_n}(x) \to \varphi_{\mu_\infty}(x)$, i.e., $\varphi_{\mu_n} \to \varphi_{\mu_\infty}$ pointwise. Similarly, for $x, y \in G_j$,

$$\begin{aligned}
\lim_{j\to\infty} \int_{K_j} \varphi_{\mu_\infty}(xky)\, dk &= \lim_{j\to\infty} \left(\int_{K_j} \varphi_{\mu_j}(xky)\, dk + \int_{K_j} \langle e_j^\perp, \pi_j(xky) e_j^\perp\rangle\, dk \right) \\
&= \lim_{j\to\infty} \varphi_{\mu_j}(x)\varphi_{\mu_j}(y) + \lim_{j\to\infty} \int_{K_j} \langle e_j^\perp, \pi_j(xky) e_j^\perp\rangle\, dk \\
&= \varphi_{\mu_\infty}(x)\varphi_{\mu_\infty}(y)
\end{aligned}$$

because
$$\left|\int_{K_j} \langle e_j^\perp, \pi_j(xky)e_j^\perp \, dk|\right| \leqq \|e_j^\perp\|^2 \int_{K_j} dk = \|e_j^\perp\|^2 \to 0\,. \qquad \square$$

The definition and construction of spherical functions in the infinite rank case remains to be clarified.

8. The Finite Rank Case

In this section we discuss the case of the finite rank Grassmannian in more detail. In this case we can assume that $\mathfrak{a}_n = \mathfrak{a}$ is fixed for all n. Then $\Sigma_n = \Sigma$ is fixed, and only the root multiplicities change as n grows.

We start with the case of the sphere. Let $X_n = S^n = SO(n+1)/SO(n)$ where the inclusions are given by $S^n \hookrightarrow S^{n+1}\,, \quad u \mapsto (u, 0)$ and
$$SO(n) \hookrightarrow SO(n+1)\,, \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathcal{H}_n(k)$ be the space of homogeneous harmonic polynomials on $\mathbb{R}^{n+1}$ of degree k with inner product
$$\langle p, q\rangle = \tfrac{1}{k!}\partial_{\overline{q}}(p)(0)\,,$$
and let $\pi_{k,n}$ denote the natural representation of $G_n = SO(n+1)$ on $\mathcal{H}_n(k)$. Then $\pi_{k,n}$ is a spherical representation of G_n and every spherical representation is constructed this way. Clearly $\mathcal{H}_n(k) \subset \mathcal{H}_m(k)$ if $n \leqq m$ and the natural inclusion is an isometry. We can take $u(x) = (x_1 - ix_2)^k$ as a highest weight vector. It is independent of n. Use polar coordinates $\cos(\theta)e_1 + \sin(\theta)u$, $u \in S^{n-1}$. Then the K_n-invariant functions corresponds to functions of one variable $P(\cos\theta)$. Using the radial component of the Laplacian on S^n, the spherical function associated to $\mathcal{H}_n(k)$ is a solution to the initial value problem
$$\begin{aligned}\left(\frac{d^2}{d\theta^2} + (n-1)\cot(\theta)\frac{d}{d\theta}\right) P_{n,k}(\cos\theta) &= -k(k+n-1)P_{n,k}(\cos\theta)\\ P_{n,k}(1) &= 1\,.\end{aligned}$$
Putting $t = \cos(\theta)$ and dividing by n,
$$\left(\frac{(1-t^2)}{n}\frac{d^2}{dt^2} - t\frac{d}{dt}\right) p_{n,k}(t) = -\left(\frac{k(k-1)}{n} + k\right) p_{n,k}(t)\,, \quad p_{n,k}(1) = 1$$
Letting $n \to \infty$ we see that the corresponding spherical function φ_∞ is a solution to the first order differential equation
$$t\frac{d}{dt}p_{\infty,k}(t) = kp_{\infty,k}(t)\,, \quad p_{\infty,k}(1) = 1$$
Thus $\varphi_{\infty,k}(x) = x_1^k$. According to [2], p. 11, every K_∞-spherical function on G_∞ is given by
$$g \mapsto \langle e_1, g(e_1)\rangle^k \text{ for some integral } k \geqq 0\,.$$
Thus all spherical functions or G_∞ are constructed as in our limit theorem. In particular, all irreducible spherical representations are obtained by the limit construction. The question is whether

that is also the case for the other finite rank Grassmannians. We have been informed that this question has been answered affirmative in [13]: Every spherical function in the sense of Olshanskii is a limit of spherical functions on X_n obtained by letting the root multiplicities go to infinity. Combining our construction with this result gives the following theorem, which in principle states that the theory of highest weights remains valid for the finite rank case.

Theorem 8.1. *Assume that the rank of X_∞ is finite. Let π be an unitary irreducible K_∞-spherical representation of G_∞. Then there exists $\mu \in \Lambda^+$ such that $\pi \simeq \pi_{\mu,\infty}$. In particular the set of equivalence classes of unitary K_∞-spherical representation of G_∞ is isomorphic to Λ^+.*

This gives us in particular a natural embedding of V_{μ_∞} into $C_b(X_\infty)$, the space of bounded continuous functions on X_∞ by $u \mapsto \langle v, \pi_{\mu_\infty}(x)e_{\mu_\infty}\rangle$. We can then ask the question: If (π, V) is a unitary irreducible representation of G_∞ with $V \subset C_b(X_\infty)$ does there exists a $\mu \in \Lambda^+$ such that $(\pi, V) \simeq (\pi_{\mu,\infty}, V_{\mu,\infty})$?

It is also natural to ask what happens in the infinite rank case. For that we would like to point the following out. Fix $\mu_n \in \Lambda^+$ and $e_n^* \in (V_n^*)^{K_n}$, $\|e_n\| = 1$. Then construct the inductive sequence (π_m, V_m) as before. Let $e_m^* \in (V_m^*)^{K_m}$ be so that the projection of e_m^* onto V_n^* is our fixed K_n-invariant functional e_n^*. Then the sequence $\{e_m^*\}$ defines an element $e_\infty^* \in \varprojlim V_m^* \simeq (\varinjlim V_m)^*$, where the limit is now with respect to the inductive/projective lmit topology in not in the category of Hilbert spaces. In particular, e_∞^* defines a linear form on $\bigcup V_n$ given by $u \mapsto \langle u, e_n^*\rangle$ if $u \in V_n$. It therefore defines a linear G_∞-equivariant embedding of $\bigcup V_n$ into $C_b(X_\infty)$.

Restricting π_{μ_∞} to G_n gives a unitary representation of G_n. Let $V_{\mu_\infty}^{n\infty}$ denote the space of smooth vectors for this representation and let

$$V_{\mu_\infty}^\infty = \bigcap_n V_{\mu_\infty}^{n\infty} .$$

Then $V_{\mu_\infty}^\infty$ is a locally convex topological vector space in the usual way. A continuous linear form $\nu : V_{\mu_\infty}^\infty \to \mathbb{C}$ is a *distribution vector*. We denote the space of distribution vectors by $V_{\mu_\infty}^{-\infty}$. The question now is whether $e_\infty^* : v \mapsto \langle v, e_\infty^*\rangle$ is a distribution vector.

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Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.

E-mail address: `mdawso5@math.lsu.edu`

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.

E-mail address: `olafsson@math.lsu.edu`

Department of Mathematics, University of California, Berkeley, CA 94720-3840, U.S.A.

E-mail address: `jawolf@math.berkeley.edu`